Derivative Formula and Harnack Inequality for Degenerate Functional SDEs*

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Abstract

By constructing successful couplings, the derivative formula, gradient estimates and Harnack inequalities are established for the semigroup associated with a class of degenerate functional stochastic differential equations.

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1 Introduction

In recent years, the coupling argument developed in [1] for establishing dimension-free Harnack inequality in the sense of [13] has been intensively applied to the study of Markov semigroups associated with a number of stochastic (partial) differential equations, see e.g. [3, 4, 6, 7, 8, 9, 14, 16, 18, 19, 20, 22] and references within. In particular, the Harnack inequalities have been established in [4, 19] for a class of non-degenerate functional stochastic differential equations (SDEs), while the (Bismut-Elworthy-Li type) derivative formula and applications have been investigated in [5] for a class of degenerate SDEs (see also [21, 23] for the study by using Malliavin calculus). The aim of this paper is to establish the derivative formula and (log-)Harnack inequalities for degenerate functional SDEs. The derivative formula implies explicit gradient estimates of the associated semigroup, while a number of

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applications of the (log-)Harnack inequalities have been summarized in [17, §4.2] on heat kernel estimates, entropy-cost inequalities, characterizations of invariant measures and contractivity properties of the semigroup.

Let $m \in \mathbb{Z}_+$ and $d \in \mathbb{N}$. Denote $\mathbb{R}^{m+d} = \mathbb{R}^m \times \mathbb{R}^d$, where $\mathbb{R}^m = \{0\}$ when $m = 0$. For $r_0 > 0$, let $\mathscr{C} := C([-r_0, 0]; \mathbb{R}^{m+d})$ be the space of continuous functions from $[-r_0, 0]$ into $\mathbb{R}^{m+d}$, which is a Banach space with the uniform norm $\|\cdot\|_\infty$. Consider the following functional SDE on $\mathbb{R}^{m+d}$:

E1 (1.1)
$$\begin{cases} \mathrm{d}X(t) = \{AX(t) + MY(t)\}\mathrm{d}t, \\ \mathrm{d}Y(t) = \{Z(X(t), Y(t)) + b(X_t, Y_t)\}\mathrm{d}t + \sigma \mathrm{d}B(t), \end{cases}$$

where $B(t)$ is a d-dimensional Brownian motion, σ is an invertible $d \times d$-matrix, A is an $m \times m$-matrix, M is an $m \times d$-matrix, $Z : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}^d$ and $b : \mathscr{C} \to \mathbb{R}^d$ are locally Lipschitz continuous (i.e. Lipschitzian on compact sets), $(X_t, Y_t)_{t \geq 0}$ is a process on $\mathscr{C}$ with $(X_t, Y_t)(\theta) := (X(t+\theta), Y(t+\theta)), \theta \in [-r_0, 0]$. We assume that there exists an integer number $0 \leq k \leq m-1$ such that

RR (1.2)
$$\mathrm{Rank}[M, AM, \cdots, A^k M] = m.$$

When $m = 0$ this condition automatically holds by convention. Note that when $m \geq 1$, this rank condition holds for some $k > m-1$ if and only if it holds for $k = m-1$.

Let $\nabla, \nabla^{(1)}$ and $\nabla^{(2)}$ denote the gradient operators on $\mathbb{R}^{m+d}, \mathbb{R}^m$ and $\mathbb{R}^d$ respectively, and let

$$\begin{aligned} Lf(x,y) :=& \langle Ax + My, \nabla^{(1)} f(x,y)\rangle + \langle Z(x,y), \nabla^{(2)} f(x,y)\rangle \\ &+ \frac{1}{2}\sum_{i,j=1}^d (\sigma\sigma^*)_{ij} \frac{\partial^2}{\partial y_i \partial y_j} f(x,y), \quad (x,y) \in \mathbb{R}^{m+d}, f \in C^2(\mathbb{R}^{m+d}). \end{aligned}$$

Since both Z and b are locally Lipschitz continuous, due to [12] the equation (1.1) has a unique local solution for any initial data $(X_0, Y_0) \in \mathscr{C}$. To ensure the non-explosion and further regular properties of the solution, we make use of the following assumptions:

(A) *There exist constants $\lambda, l > 0$ and $W \in C^2(\mathbb{R}^{m+d})$ of compact level sets with $W \geq 1$ such that*

(A1) $LW \leq \lambda W, \quad |\nabla^{(2)} W| \leq \lambda W;$

(A2) $\langle b(\xi), \nabla^{(2)} W(\xi(0))\rangle \leq \lambda \|W(\xi)\|_\infty, \quad \xi \in \mathscr{C};$

(A3) $|Z(z) - Z(z')| \leq \lambda |z - z'| W(z')^l, \quad z, z' \in \mathbb{R}^{m+d}, |z - z'| \leq 1;$

(A4) $|b(\xi) - b(\xi')| \leq \lambda \|\xi - \xi'\|_\infty \|W(\xi')\|_\infty^l, \ \xi, \xi' \in \mathscr{C}, \|\xi - \xi'\|_\infty \leq 1.$

Comparing with the framework investigated in [5, 23], where $b = 0, A = 0$ and $\mathrm{Rank}[M] = m$ are assumed, the present model is more general and the segment process we are going to investigate is an infinite-dimensional Markov process. On the other hand, unlike in [5] where the condition $|\nabla^{(2)} W| \le \lambda W$ is not used, in the present setting this condition seems essential in order to derive moment estimates of the segment process (see the proof of Lemma 2.1 below). Moreover, if $|\nabla W| \le cW$ holds for some constant $c > 0$, then $(A3)$ and $(A4)$ hold for some $\lambda > 0$ if and only if there exists a constant $\lambda' > 0$ such that $|\nabla Z| \le \lambda' W^l$ and $|\nabla b| \le \lambda' \|W\|_\infty^l$ holds on $\mathbb{R}^{m+d}$ and $\mathscr{C}$ respectively.

It is easy to see that **(A)** holds for $W(z) = 1 + |z|^2$, $l = 1$ and some constant $\lambda > 0$ provided that Z and b are globally Lipschitz continuous on $\mathbb{R}^{m+d}$ and $\mathscr{C}$ respectively. It is clear that $(A1)$ and $(A2)$ imply the non-explosion of the solution (see Lemma 2.1 below). In this paper we aim to investigate regularity properties of the Markov semigroup associated with the segment process:

$$P_t f(\xi) = \mathbb{E}^\xi f(X_t, Y_t), \quad f \in \mathscr{B}_b(\mathscr{C}), \xi \in \mathscr{C},$$

where $\mathscr{B}_b(\mathscr{C})$ is the class of all bounded measurable functions on $\mathscr{C}$ and $\mathbb{E}^\xi$ stands for the expectation for the solution starting at the point $\xi \in \mathscr{C}$. When $m = 0$ we have $X_t \equiv 0$ and $\mathscr{C} = \{0\} \times \mathscr{C}_2 \equiv \mathscr{C}_2 := C([-r_0, 0]; \mathbb{R}^d)$, so that $P_t f$ can be simply formulated as $P_t f(\xi) = \mathbb{E}^\xi f(Y_t)$ for $f \in \mathscr{B}_b(\mathscr{C}_2), \xi \in \mathscr{C}_2$. Thus, (1.1) also includes non-degenerate functional SDEs. For any $h = (h_1, h_2) \in \mathscr{C}$ and $z \in \mathbb{R}^{m+d}$, let ∇_h and ∇_z be the directional derivatives along h and z respectively. The following result provides an explicit derivative formula for $P_T, T > r_0$.

T1.1 **Theorem 1.1.** *Assume* **(A)** *and let $T > r_0$. Let $v : [0, T] \to \mathbb{R}$ and $\alpha : [0, T] \to \mathbb{R}^m$ be Lipschitz continuous such that $v(0) = 1, \alpha(0) = 0, v(s) = 0, \alpha(s) = 0$ for $s \ge T - r_0$, and*

LL (1.3)
$$h_1(0) + \int_0^t \mathrm{e}^{-sA} M\phi(s)\mathrm{d}s = 0, \quad t \ge T - r_0,$$

where $\phi(s) := v(s)h_2(0) + \alpha(s)$. Then for any $h = (h_1, h_2) \in \mathscr{C}$ and $f \in \mathscr{B}_b(\mathscr{C})$,

Bis (1.4)
$$\nabla_h P_T f(\xi) = \mathbb{E}^\xi \bigg\{ f(X_T, Y_T) \int_0^T \big\langle N(s), (\sigma^*)^{-1} \mathrm{d}B(s) \big\rangle \bigg\}, \quad \xi \in \mathscr{C}$$

holds for

$$N(s) := (\nabla_{\Theta(s)} Z)(X(s), Y(s)) + (\nabla_{\Theta_s} b)(X_s, Y_s) - v'(s)h_2(0) - \alpha'(s), \quad s \in [0, T],$$

where

$$\Theta(s) = (\Theta^{(1)}(s), \Theta^{(2)}(s)) := \begin{cases} h(s), & \text{if } s \le 0, \\ \big(\mathrm{e}^{As} h_1(0) + \int_0^s \mathrm{e}^{(s-r)A} M\phi(r)\mathrm{d}r,\ \phi(s)\big), & \text{if } s > 0. \end{cases}$$

A simple choice of v is

$$v(s) = \frac{(T - r_0 - s)^+}{T - r_0}, \quad s \geq 0.$$

To present a specific choice of α, let

$$Q_t := \int_0^t \frac{s(T - r_0 - s)^+}{(T - r_0)^2} \mathrm{e}^{-sA} M M^* \mathrm{e}^{-sA^*} \mathrm{d}s, \quad t > 0.$$

According to [11] (see also [21, Proof of Theorem 4.2(1)]), when $m \geq 1$ the matrix Q_t is invertible with

QQ (1.5)
$$\|Q_t^{-1}\| \leq c(T - r_0)(t \wedge 1)^{-2(k+1)}, \quad t > 0$$

for some constant $c > 0$.

C1.2 **Corollary 1.2.** *Assume* **(A)** *and let* $T > r_0$. *Then* (1.4) *holds for* $v(s) = \frac{(T-r_0-s)^+}{T-r_0}$ *and*

$$\alpha(s) = -\frac{s(T - r_0 - s)^+}{(T - r_0)^2} M^* \mathrm{e}^{-sA^*} Q_{T-r_0}^{-1} \bigg(h_1(0) + \int_0^{T-r_0} \frac{(T - r_0 - r)^+}{T - r_0} \mathrm{e}^{-rA} M h_2(0) \mathrm{d}r \bigg),$$

where by convention $M = 0$ *(hence,* $\alpha = 0$*) if* $m = 0$.

The following gradient estimates are direct consequences of Theorem 1.1.

C1.3 **Corollary 1.3.** *Assume* **(A)**. *Then:*

(1) *There exists a constant* $C \in (0, \infty)$ *such that*

$$\begin{aligned} |\nabla_h P_T f(\xi)| \leq C \sqrt{P_T f^2(\xi)} \bigg\{ & |h(0)| \Big(1 + \frac{\|M\|}{(T - r_0)^{2k+1} \wedge 1} \Big) \\ & + \|W(\xi)\|_\infty^l \sqrt{T \wedge (1 + r_0)} \Big(\|h\|_\infty + \frac{\|M\| \cdot |h(0)|}{(T - r_0)^{2k+1} \wedge 1} \Big) \bigg\} \end{aligned}$$

holds for all $T > r_0, \xi, h \in \mathscr{C}$ *and* $f \in \mathscr{B}_b(\mathscr{C})$;

(2) *Let* $|\nabla^{(2)} W|^2 \leq \delta W$ *hold for some constant* $\delta > 0$. *If* $l \in [0, 1/2)$ *then there exists a constant* $C \in (0, \infty)$ *such that*

$$\begin{aligned} |\nabla_h P_T f(\xi)| \leq\; & r \big\{ P_T f \log f - (P_T f) \log P_T f \big\}(\xi) \\ & + \frac{C P_T f(\xi)}{r} \bigg\{ |h(0)|^2 \bigg(\frac{1}{(T - r_0) \wedge 1} + \frac{\|M\|^2}{\{(T - r_0) \wedge 1\}^{4k+3}} \bigg) \\ & + \|h\|_\infty^2 \|W(\xi)\|_\infty + \bigg(\|h\|_\infty^2 + \frac{|h(0)|^2 \|M\|^2}{\{(T - r_0) \wedge 1\}^{4k+2}} \bigg)^{\frac{1}{1-2l}} \bigg(\frac{r^2}{\|h\|_\infty^2} \vee 1 \bigg)^{\frac{2l}{1-2l}} \bigg\} \end{aligned}$$

holds for all $r > 0, T > r_0, \xi, h \in \mathscr{C}$ *and positive* $f \in \mathscr{B}_b(\mathscr{C})$;

(3) *Let $|\nabla^{(2)}W|^2 \le \delta W$ hold for some constant $\delta > 0$. If $l = \frac{1}{2}$ then there exist constants $C, C' \in (0,\infty)$ such that*

$$\begin{aligned}|\nabla_h P_T f(\xi)| \le & r\big\{P_T f \log f - (P_T f)\log P_T f\big\}(\xi) \\ & + \frac{C P_T f(\xi)}{r}\Big\{|h(0)|^2\Big(\frac{1}{(T-r_0)\wedge 1} + \frac{\|M\|^2}{\{(T-r_0)\wedge 1\}^{4k+3}}\Big) \\ & + \|W(\xi)\|_\infty\Big(\|h\|_\infty^2 + \frac{\|M\|^2|h(0)|^2}{\{(T-r_0)\wedge 1\}^{4k+2}}\Big)\Big\}\end{aligned}$$

holds for

$$r \ge C'\Big(\|h\|_\infty + \frac{\|M\|\cdot|h(0)|}{\{(T-r_0)\wedge 1\}^{2k+1}}\Big),$$

all $T > r_0, \xi, h \in \mathscr{C}$ and positive $f \in \mathscr{B}_b(\mathscr{C})$.

When $m = 0$ the above assertions hold with $\|M\| = 0$.

According to [2], the entropy gradient estimate implies the Harnack inequality with power, we have the following result which follows immediately from Corollary 1.3 (2) and [5, Proposition 4.1]. Similarly, Corollary 1.3 (3) implies the same type Harnack inequality for smaller $\|h\|_\infty$ comparing to $T - r_0$.

C1.4 **Corollary 1.4.** *Assume* **(A)** *and let $|\nabla^{(2)}W|^2 \le \delta W$ hold for some constant $\delta > 0$. If $l \in [0, \frac{1}{2})$ then there exists a constant $C \in (0,\infty)$ such that*

$$\begin{aligned}(P_T f)^p(\xi+h) \le & P_T f^p(\xi)\exp\bigg[\frac{Cp}{p-1}\bigg\{\|h\|_\infty^2\int_0^1 \|W(\xi+sh)\|_\infty \mathrm{d}s \\ & + \Big(\|h\|_\infty^2 + \frac{\|M\|^2|h(0)|^2}{\{(T-r_0)\wedge 1\}^{4k+2}}\Big)^{\frac{1}{1-2l}}\Big(\frac{(p-1)^2}{\|h\|_\infty^2}\vee 1\Big)^{\frac{2l}{1-2l}}\bigg\}\bigg]\end{aligned}$$

holds for all $T > r_0, p > 1, \xi, h \in \mathscr{C}$ and positive $f \in \mathscr{B}_b(\mathscr{C})$. If $m = 0$ then the assertion holds for $\|M\| = 0$.

Finally, we consider the log-Harnack inequality introduced in [10, 15]. To this end, as in [5], we slightly strengthen $(A3)$ and $(A4)$ as for follows: there exists an increasing function U on $[0,\infty)$ such that

$(A3')$ $|Z(z) - Z(z')| \le \lambda|z - z'|\big\{W(z')^l + U(|z-z'|)\big\}, \quad z, z' \in \mathbb{R}^{m+d}$;

$(A4')$ $|b(\xi) - b(\xi')| \le \lambda\|\xi - \xi'\|_\infty\big\{\|W(\xi')\|_\infty^l + U(\|\xi-\xi'\|_\infty)\big\}$, $\xi, \xi' \in \mathscr{C}$.

Obviously, if

$$W(z)^l \le c\{W(z')^l + U(|z-z'|)\}, \quad z, z' \in \mathbb{R}^{m+d}$$

holds for some constant $c > 0$, then $(A3)$ and $(A4)$ imply $(A3')$ and $(A4')$ respectively with possibly different λ.

T1.5 **Theorem 1.5.** *Assume* $(A1),(A2),(A3')$ *and* $(A4')$. *Then there exists a constant* $C \in (0,\infty)$ *such that for any positive* $f \in \mathscr{B}_b(\mathscr{C}), T > r_0$ *and* $\xi, h \in \mathscr{C}$,

$$P_T \log f(\xi+h) - \log P_T f(\xi) \le C\bigg\{\bigg[\|W(\xi+h)\|_\infty^{2l} + U^2\Big(C\|h\|_\infty + \frac{C\|M\|\cdot|h(0)|}{(T-r_0)\wedge 1}\Big)\bigg]\|h\|_\infty^2 + \frac{|h(0)|^2}{(T-r_0)\wedge 1} + \frac{\|M\|^2|h(0)|^2}{\{(T-r_0)\wedge 1\}^{4k+3}}\bigg\}.$$

If $m=0$ *then the assertion holds for* $\|M\|=0$.

For applications of the Harnack and log-Harnack inequalities we are referred to [17, §4.2]. The remainder of the paper is organized as follows: Theorem 1.1 and Corollary 1.2 are proved Section 2, while Corollary 1.3 and Theorem 1.5 are proved in Section 3; in Section 4 the assumption **(A)** is weakened for the discrete time delay case, and two examples are presented to illustrate our results.

2 Proofs of Theorem 1.1 and Corollary 1.2

lem1 **Lemma 2.1.** *Assume* $(A1)$ *and* $(A2)$. *Then for any* $k>0$ *there exists a constant* $C>0$ *such that*

$$\mathbb{E}^\xi \sup_{-r_0\le s\le t} W(X(s),Y(s))^k \le 3\|W(\xi)\|_\infty^k \mathrm{e}^{Ct}, \quad t\ge 0,\ \xi\in\mathscr{C}$$

holds. Consequently, the solution is non-explosive.

Proof. For any $n\ge 1$, let

$$\tau_n := \inf\{t\in[0,T] : |X(t)|+|Y(t)|\ge n\}.$$

Moreover, let

$$\ell(s) := W(X,Y)(s), \quad s\ge -r_0.$$

By the Itô formula and using the first inequality in $(A1)$ and $(A2)$ we may find a constant $C_1>0$ such that

W2 (2.1)
$$\begin{aligned}\ell(t\wedge\tau_n)^k &= \ell(0)^k + k\int_0^{t\wedge\tau_n}\ell(s)^{k-1}\langle\nabla^{(2)}W(X,Y)(s),\sigma \mathrm{d}B(s)\rangle\\ &+ k\int_0^{t\wedge\tau_n}\ell(s)^{k-1}\Big\{LW(X,Y)(s) + \big\langle b(X_s,Y_s),\nabla^{(2)}W(X,Y)(s)\big\rangle\\ &+\frac{1}{2}(k-1)\ell(s)^{-1}|\sigma^*\nabla^{(2)}W(X,Y)(s)|^2\Big\}\mathrm{d}s\\ &\le l(0)^k + k\int_0^{t\wedge\tau_n}\ell(s)^{k-1}\langle\nabla^{(2)}W(X,Y)(s),\sigma\mathrm{d}B(s)\rangle + C_1\int_0^{t\wedge\tau_n}\sup_{r\in[-r_0,s]}\ell(r)^k\mathrm{d}s.\end{aligned}$$

Noting that by the second inequality in $(A1)$ and the Burkholder-Davis-Gundy inequality we obtain

$$k\mathbb{E}^\xi \sup_{s\in[0,t]} \Big| \int_0^{s\wedge\tau_n} \ell(r)^{k-1} \langle \nabla^{(2)} W(X,Y)(s), \sigma \mathrm{d}B(r)\rangle \Big| \le C_2 \mathbb{E}^\xi \Big(\int_0^t \ell(s\wedge\tau_n)^{2k} \mathrm{d}s\Big)^{1/2}$$
$$\le C_2\mathbb{E}^\xi \Big\{ \Big(\sup_{s\in[0,t]} \ell(s\wedge\tau_n)^k\Big)^{1/2} \Big(\int_0^t \ell(s\wedge\tau_n)^k \mathrm{d}s\Big)^{1/2}\Big\}$$
$$\le \frac{1}{2}\mathbb{E}^\xi \sup_{s\in[0,t]} \ell(s\wedge\tau_n)^k + \frac{C_2^2}{2}\mathbb{E}^\xi \int_0^t \sup_{r\in[0,s]} \ell(r\wedge\tau_n)^k \mathrm{d}s$$

for some constant $C_2 > 0$. Combining this with (2.1) and noting that $(X_0, Y_0) = \xi$, we conclude that there exists a constant $C > 0$ such that

$$\mathbb{E}^\xi \sup_{-r_0\le s\le t} \ell(s\wedge\tau_n)^k \le 3\|W(\xi)\|_\infty^k + C\mathbb{E}^\xi \int_0^t \sup_{s\in[-r_0,t]} \ell(s)^k \mathrm{d}s, \quad t\ge 0.$$

Due to the Gronwall lemma this implies that

$$\mathbb{E}^\xi \sup_{-r_0\le s\le t} \ell(s\wedge\tau_n)^k \le 3\|W(\xi)\|_\infty^k \mathrm{e}^{Ct}, \quad t\ge 0, n\ge 1.$$

Consequently, we have $\tau_n \uparrow \infty$ as $n\uparrow\infty$, and thus the desired inequality follows by letting $n\to\infty$. □

To establish the derivative formula, we first construct couplings for solutions starting from ξ and $\xi+\varepsilon h$ for $\varepsilon\in(0,1]$, then let $\varepsilon\to 0$. For fixed $\xi=(\xi_1,\xi_2), h=(h_1,h_2)\in\mathscr{C}$, let $(X(t),Y(t))$ solve (1.1) with $(X_0,Y_0)=\xi$; and for any $\varepsilon\in(0,1]$, let $(X^\varepsilon(t),Y^\varepsilon(t))$ solve the equation

E2 (2.2)
$$\begin{cases} \mathrm{d}X^\varepsilon(t) = \{AX^\varepsilon(t) + MY^\varepsilon(t)\}\mathrm{d}t,\\ \mathrm{d}Y^\varepsilon(t) = \{Z(X(t),Y(t)) + b(X_t,Y_t)\}\mathrm{d}t + \sigma\mathrm{d}B(t) + \varepsilon\{v'(t)h_2(0) + \alpha'(t)\}\mathrm{d}t \end{cases}$$

with $(X_0^\varepsilon, Y_0^\varepsilon) = \xi + \varepsilon h$. By Lemma 2.1 and (2.3) below, the solution to (2.2) is non-explosive as well.

Pro1 **Proposition 2.2.** *Let $\phi(s) := v(s)h_2(0) + \alpha(s),\ s\in[0,T]$, and the conditions of Theorem 1.1 hold. Then*

EE (2.3)
$$(X^\varepsilon(t), Y^\varepsilon(t)) = (X(t),Y(t)) + \varepsilon\Theta(t), \quad \varepsilon, t\ge 0$$

holds for

$$\Theta(t) := (\Theta^{(1)}(t), \Theta^{(2)}(t)) := \begin{cases} h(t), & \text{if } t\le 0,\\ \big(\mathrm{e}^{At}h_1(0) + \int_0^t \mathrm{e}^{(t-r)A} M\phi(r)\mathrm{d}r,\ \phi(t)\big), & \text{if } t>0. \end{cases}$$

In particular, $(X_T^\varepsilon, Y_T^\varepsilon) = (X_T, Y_T)$.

Proof. By (2.2) and noting that $v(0) = 1$ and $v(s) = 0$ for $s \geq T - r_0$, we have $Y^\varepsilon(t) = Y(t) + \varepsilon\phi(t)$ and

$$X^\varepsilon(t) = X(t) + \varepsilon \mathrm{e}^{At} h_1(0) + \varepsilon \int_0^t \mathrm{e}^{(t-s)A} M\phi(s)\mathrm{d}s, \quad t \geq 0.$$

Thus, (2.3) holds. Moreover, since $\alpha(s) = v(s) = 0$ for $s \geq T - r_0$, we have $\Theta^{(2)}(s) = \phi(s) = 0$ for $s \geq T - r_0$. Moreover, by (1.3) we have $\Theta^{(1)}(s) = 0$ for $s \geq T - r_0$. Therefore, the proof is finished. $\square$

Since according to Proposition 2.2 we have $(X^\varepsilon_T, Y^\varepsilon_T) = (X_T, Y_T)$. Noting that $(X^\varepsilon_0, Y^\varepsilon_0) = \xi + \varepsilon h$, if (2.2) can be formulated as (1.1) using a different Brownian motion, then we are able to link $P_T f(\xi)$ to $P_T f(\xi + \varepsilon h)$ and furthermore derive the derivative formula by taking derivative w.r.t. ε at $\varepsilon = 0$. To this end, let

$$\Phi^\varepsilon(s) = Z(X(s), Y(s)) - Z(X^\varepsilon(s), Y^\varepsilon(s)) + b(X_s, Y_s) - b(X^\varepsilon_s, Y^\varepsilon_s) + \varepsilon\{v'(s)h_2(0) + \alpha'(s)\}.$$

Set

$$R^\varepsilon(s) = \exp\left[- \int_0^s \langle \sigma^{-1}\Phi^\varepsilon(r), \mathrm{d}B(r)\rangle - \frac{1}{2}\int_0^s |\sigma^{-1}\Phi^\varepsilon(r)|^2 \mathrm{d}r \right],$$

and

$$B^\varepsilon(s) = B(s) + \int_0^s \sigma^{-1}\Phi^\varepsilon(r)\mathrm{d}r.$$

Then (2.2) reduces to

E2' (2.4)
$$\begin{cases} \mathrm{d}X^\varepsilon(t) = \{AX^\varepsilon(t) + MY^\varepsilon(t)\}\mathrm{d}t, \\ \mathrm{d}Y^\varepsilon(t) = \{Z(X^\varepsilon(t), Y^\varepsilon(t)) + b(X^\varepsilon_t, Y^\varepsilon_t)\}\mathrm{d}t + \sigma \mathrm{d}B^\varepsilon(t). \end{cases}$$

According to the Girsanov theorem, to ensure that $B^\varepsilon(t)$ is a Browanian motion under $\mathbb{Q}_\varepsilon := R^\varepsilon(T)\mathbb{P}$, we first prove that $R^\varepsilon(t)$ is an exponential martingale. Moreover, to obtain the derivative formula using the dominated convergence theorem, we also need $\{\frac{R^\varepsilon(T)-1}{\varepsilon}\}_{\varepsilon\in(0,1)}$ to be uniformly integrable. Therefore, we will need the following two lemmas.

L2.2 **Lemma 2.3.** *Let* **(A)** *hold. Then there exists $\varepsilon_0 > 0$ such that*

$$\sup_{s\in[0,T],\varepsilon\in(0,\varepsilon_0)} \mathbb{E}[R^\varepsilon(s)\log R^\varepsilon(s)] < \infty,$$

so that for each $\varepsilon \in (0,1)$, $(R^\varepsilon(s))_{s\in[0,T]}$ is a uniformly integrable martingale.

Proof. By (2.3), there exists $\varepsilon_0 > 0$ such that

ED (2.5)
$$\varepsilon_0 |\Theta(t)| \leq 1, \quad t \in [-r_0, T].$$

For any $\varepsilon \in [0, \varepsilon_0]$, define

$$\tau_n := \inf\{t \geq 0 : |X(t)| + |Y(t)| + |X^\varepsilon(t)| + |Y^\varepsilon(t)| \geq n\},\ n \geq 1.$$

We have $\tau_n \uparrow \infty$ as $n \uparrow \infty$ due to the non-explosion. By the Girsanov theorem, the process $\{R^\varepsilon(s \wedge \tau_n)\}_{s\in[0,T]}$ is a martingale and $\{B^\varepsilon(s)\}_{s\in[0,T\wedge\tau_n]}$ is a Brownian motion under the probability measure $\mathbb{Q}_{\varepsilon,n} := R^\varepsilon(T \wedge \tau_n)\mathbb{P}$. By the definition of $R^\varepsilon(s)$ we have

2.6
$$\mathbb{E}[R^\varepsilon(s\wedge\tau_n)\log R^\varepsilon(s\wedge\tau_n)] = \mathbb{E}_{\mathbb{Q}_{\varepsilon,n}}[\log R^\varepsilon(s\wedge\tau_n)] \le \frac{1}{2}\mathbb{E}_{\mathbb{Q}_{\varepsilon,n}}\int_0^{T\wedge\tau_n} |\sigma^{-1}\Phi^\varepsilon(r)|^2\mathrm{d}r. \tag{2.6}$$

By (2.5), $(A3)$ and $(A4)$,

2.7
$$|\sigma^{-1}\Phi^\varepsilon(s)|^2 \le c\varepsilon^2\|W(X_s^\varepsilon, Y_s^\varepsilon)\|_\infty^{2l}, \tag{2.7}$$

holds for some constant c independent of ε. By the weak uniqueness of the solution to (1.1) and (2.4), the distribution of $(X^\varepsilon(s), Y^\varepsilon(s))_{s\in[0,T\wedge\tau_n]}$ under $\mathbb{Q}_{\varepsilon,n}$ coincides with that of the solution to (1.1) with $(X_0, Y_0) = \xi + \varepsilon h$ up to time $T \wedge \tau_n$, we therefore obtain from Lemma 2.1 that

$$\mathbb{E}[R^\varepsilon(s\wedge\tau_n)\log R^\varepsilon(s\wedge\tau_n)] \le c\|W(\xi+\varepsilon h)\|_\infty^{2l}\int_0^T \mathrm{e}^{Ct}\mathrm{d}t < \infty, \quad n \ge 1, \varepsilon \in (0,\varepsilon_0).$$

Then the required assertion follows by letting $n \to \infty$. □

L2.3 **Lemma 2.4.** *If* **(A)** *holds, then there exists* $\varepsilon_0 > 0$ *such that*

$$\sup_{\varepsilon\in(0,\varepsilon_0)} \mathbb{E}\left(\frac{R^\varepsilon(T)-1}{\varepsilon}\log\frac{R^\varepsilon(T)-1}{\varepsilon}\right) < \infty.$$

Moreover,

y2
$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{R^\varepsilon(T)-1}{\varepsilon} = \\ &\int_0^T \big\langle(\nabla_{\Theta(s)}Z)(X(s),Y(s)) + (\nabla_{\Theta_s}b)(X_s,Y_s) - v'(s)h_2(0) - \alpha'(s), (\sigma^*)^{-1}\mathrm{d}B(s)\big\rangle.\end{aligned} \tag{2.8}$$

Proof. Let ε_0 be such that (2.5) holds. Since (2.8) is a direct consequence of (2.3) and the definition of $R^\varepsilon(T)$, we only prove the first assertion. By [5] we know that

$$\frac{R^\varepsilon(T)-1}{\varepsilon}\log\frac{R^\varepsilon(T)-1}{\varepsilon} \le 2R^\varepsilon(T)\left(\frac{\log R^\varepsilon(T)}{\varepsilon}\right)^2.$$

Since due to Lemma 2.3 $\{B^\varepsilon(t)\}_{t\in[0,T]}$ is a Brownian motion under the probability measure $\mathbb{Q}_\varepsilon := R^\varepsilon(T)\mathbb{P}$, and since

$$\begin{aligned}\log R^\varepsilon(T) &= -\int_0^T \langle\sigma^{-1}\Phi^\varepsilon(r), \mathrm{d}B(r)\rangle - \frac{1}{2}\int_0^T |\sigma^{-1}\Phi^\varepsilon(r)|^2\mathrm{d}r \\ &= -\int_0^T \langle\sigma^{-1}\Phi^\varepsilon(r), \mathrm{d}B^\varepsilon(r)\rangle + \frac{1}{2}\int_0^T |\sigma^{-1}\Phi^\varepsilon(r)|^2\mathrm{d}r,\end{aligned}$$

it follows from (2.7) that

$$
\begin{aligned}
&\mathbb{E}\bigg(\frac{R^\varepsilon(T)-1}{\varepsilon}\log\frac{R^\varepsilon(T)-1}{\varepsilon}\bigg)\le\mathbb{E}\bigg(2R^\varepsilon(T)\Big(\frac{\log R^\varepsilon(T)}{\varepsilon}\Big)^2\bigg)=2\mathbb{E}_{\mathbb{Q}_\varepsilon}\Big(\frac{\log R^\varepsilon(T)}{\varepsilon}\Big)^2\\
&\le\frac{4}{\varepsilon^2}\mathbb{E}_{\mathbb{Q}_\varepsilon}\bigg(\int_0^T\langle\sigma^{-1}\Phi^\varepsilon(r),\mathrm{d}B^\varepsilon(r)\rangle\bigg)^2+\frac{1}{\varepsilon^2}\mathbb{E}_{\mathbb{Q}_\varepsilon}\bigg(\int_0^T|\sigma^{-1}\Phi^\varepsilon(r)|^2\mathrm{d}r\bigg)^2\\
&\le\frac{4}{\varepsilon^2}\int_0^T\mathbb{E}_{\mathbb{Q}_\varepsilon}|\sigma^{-1}\Phi^\varepsilon(r)|^2\mathrm{d}r+\frac{T}{\varepsilon^2}\int_0^T\mathbb{E}_{\mathbb{Q}_\varepsilon}|\sigma^{-1}\Phi^\varepsilon(r)|^4\mathrm{d}r\\
&\le c\int_0^T\mathbb{E}_{\mathbb{Q}_\varepsilon}\|W(X_r^\varepsilon,Y_r^\varepsilon)\|_\infty^{4l}\mathrm{d}r
\end{aligned}
$$

holds for some constant $c>0$. As explained in the proof of Lemma 2.3 the distribution of $(X_s^\varepsilon,Y_s^\varepsilon)_{s\in[0,T]}$ under $\mathbb{Q}_\varepsilon$ coincides with that of the segment process of the solution to (1.1) with $(X_0,Y_0)=\xi+\varepsilon h$, the first assertion follows by Lemma 2.1. □

Proof of Theorem 1.1. Since Lemma 2.3, together with the Girsanov theorem, implies that $\{B^\varepsilon(s)\}_{s\in[0,T]}$ is a Brownian motion with respect to $\mathbb{Q}_\varepsilon:=R^\varepsilon(T)\mathbb{P}$, by (2.4) and $(X_T,Y_T)=(X_T^\varepsilon,Y_T^\varepsilon)$ we obtain

y1 (2.9)
$$P_Tf(\xi+\varepsilon h)=\mathbb{E}_{\mathbb{Q}_\varepsilon}f(X_T^\varepsilon,Y_T^\varepsilon)=\mathbb{E}\{R^\varepsilon(T)f(X_T,Y_T)\}.$$

Thus,

$$P_Tf(\xi+\varepsilon h)-P_Tf(\xi)=\mathbb{E}R^\varepsilon(T)f(X_T,Y_T)-\mathbb{E}f(X_T,Y_T)=\mathbb{E}[(R^\varepsilon(T)-1)f(X_T,Y_T)].$$

Combining this with Lemma 2.4 and using the dominated convergence theorem, we arrive at

$$
\begin{aligned}
\nabla_hP_Tf(\xi,\eta)&=\lim_{\varepsilon\to0}\frac{P_Tf(\xi+\varepsilon h)-P_Tf(\xi)}{\varepsilon}=\lim_{\varepsilon\to0}\frac{\mathbb{E}[(R^\varepsilon(T)-1)f(X_T,Y_T)]}{\varepsilon}\\
&=\mathbb{E}\bigg\{f(X_T,Y_T)\int_0^T\big\langle N(s),(\sigma^*)^{-1}\mathrm{d}B(s)\big\rangle\bigg\}.
\end{aligned}
$$

□

Proof of Corollary 1.2. It suffices to verify (1.3) for the specific v and α. Since when $m=0$ we have $h_1=M=0$ so that (1.3) trivially holds, we only consider $m\ge1$. In this case, (1.3) is satisfied since according to the definition of $\phi(s)$ and $\alpha(s)$ we have for $t\ge T-r_0$,

$$
\begin{aligned}
&\int_0^t\mathrm{e}^{-sA}M\phi(s)\mathrm{d}s=\int_0^{T-r_0}\mathrm{e}^{-sA}M\phi(s)\mathrm{d}s\\
&=\int_0^{T-r_0}v(s)\mathrm{e}^{-sA}Mh_2(0)\mathrm{d}s-Q_{T-r_0}Q_{T-r_0}^{-1}\bigg(h_1(0)+\int_0^{T-r_0}v(s)\mathrm{e}^{-sA}Mh_2(0)\mathrm{d}s\bigg)\\
&=-h_1(0).
\end{aligned}
$$

□

3 Proofs of Corollary 1.3 and Theorem 1.5

To prove the entropy-gradient estimates in Corollary (2) and (3), we need the following simple lemma which seems new and might be interesting by itself.

L3.1 **Lemma 3.1.** *Let $\ell(t)$ be a non-negative continuous semi-martingale and let $\mathscr{M}(t)$ be a continuous martingale with $\mathscr{M}(0)=0$ such that*

$$\mathrm{d}\ell(t) \le \mathrm{d}\mathscr{M}(t) + c\bar\ell_t \mathrm{d}t,$$

where $c \ge 0$ is a constant and $\bar\ell_t := \sup_{s\in[0,t]} \ell(s)$. Then

$$\mathbb{E}\exp\left[\frac{\varepsilon}{T\mathrm{e}^{1+cT}}\int_0^T \bar\ell_t \mathrm{d}t\right] \le \mathrm{e}^{\varepsilon\ell(0)+1}\big(\mathbb{E}\mathrm{e}^{2\varepsilon^2\langle\mathscr{M}\rangle(T)}\big)^{1/2}, \quad T,\varepsilon \ge 0.$$

Proof. Let $\bar{\mathscr{M}}_t := \sup_{s\in[0,t]}\mathscr{M}(t)$. We have

$$\bar{\mathscr{M}}_t + c\int_0^t \bar\ell_s \mathrm{d}s \ge \bar\ell_t - \ell(0).$$

Thus,

$$\begin{aligned}
\frac{\ell_T}{\mathrm{e}^{1+cT}} - \ell(0) &\le \frac{\bar{\mathscr{M}}_T + c\int_0^T \bar\ell_t \mathrm{d}t}{\mathrm{e}^{1+cT}} - \big(1-\mathrm{e}^{-(1+cT)}\big)\ell(0)\\
&= \int_0^T \mathrm{d}\Big\{\mathrm{e}^{-(c+T^{-1})t}\Big(\bar{\mathscr{M}}_t + c\int_0^t \bar\ell_s\mathrm{d}s\Big)\Big\} - \big(1-\mathrm{e}^{-(1+cT)}\big)\ell(0)\\
&= \int_0^T \mathrm{e}^{-(T^{-1}+c)t}\mathrm{d}\bar{\mathscr{M}}_t + \int_0^T \mathrm{e}^{-(c+T^{-1})t}\Big\{c\bar\ell_t - (T^{-1}+c)\Big(\bar{\mathscr{M}}_t + c\int_0^t \bar\ell_s\mathrm{d}s\Big)\Big\}\mathrm{d}t\\
&\qquad - \big(1-\mathrm{e}^{-(1+cT)}\big)\ell(0)\\
&\le \bar{\mathscr{M}}_T + \int_0^T \mathrm{e}^{-(c+T^{-1})t}\Big\{c\bar\ell_t - (T^{-1}+c)\big(\bar\ell_t - \ell(0)\big)\Big\}\mathrm{d}t - \big(1-\mathrm{e}^{-(1+cT)}\big)\ell(0)\\
&\le \bar{\mathscr{M}}_T - \frac{1}{T\mathrm{e}^{1+cT}}\int_0^T \bar\ell_t\mathrm{d}t.
\end{aligned}$$

Combining this with

$$\mathbb{E}\mathrm{e}^{\varepsilon\bar{\mathscr{M}}_t} \le \mathbb{E}\mathrm{e}^{1+\varepsilon\mathscr{M}(T)} \le \mathrm{e}\big(\mathbb{E}\mathrm{e}^{2\varepsilon^2\langle\mathscr{M}\rangle(T)}\big)^{1/2},$$

we complete the proof. □

C3.1 **Corollary 3.2.** *Assume* **(A)** *and let $|\nabla^{(2)}W|^2 \le \delta W$ hold for some constant $\delta > 0$. Then there exists a constant $c > 0$ such that*

$$\begin{aligned}
&\mathbb{E}^\xi \exp\left[\frac{1}{2\|\sigma\|^2\delta T^2\mathrm{e}^{2+2cT}}\int_0^T \|W(X_t,Y_t)\|_\infty \mathrm{d}t\right]\\
&\le \exp\left[2 + \frac{W(\xi(0))}{\|\sigma\|^2\delta T\mathrm{e}^{1+cT}} + \frac{r_0\|W(\xi)\|_\infty}{2\|\sigma\|^2\delta T^2\mathrm{e}^{2+2cT}}\right], \quad T > r_0.
\end{aligned}$$

Proof. By **(A)** and the Itô formula, there exists a constant $c > 0$ such that

$$\mathrm{d}W(X,Y)(s) \le \langle \nabla^{(2)} W(X,Y)(s), \sigma \mathrm{d}B(s)\rangle + c\|W(X_s,Y_s)\|_\infty \mathrm{d}s.$$

Let

$$\mathscr{M}(t) := \int_0^t \langle \nabla^{(2)} W(X,Y)(s), \sigma \mathrm{d}B(s)\rangle, \quad l(t) := W(X,Y)(t),$$

and let $\varepsilon = (2\|\sigma\|^2 \delta T \mathrm{e}^{1+cT})^{-1}$ such that

$$\frac{\varepsilon}{T\mathrm{e}^{1+cT}} = 2\|\sigma\|^2\varepsilon^2.$$

Then by Lemma 3.1 and $|\nabla^{(2)} W|^2 \le \delta W$, we have

$$\begin{aligned}\mathbb{E}^\xi \exp\left[\frac{\varepsilon}{T\mathrm{e}^{1+cT}}\int_0^T \bar{l}_t \mathrm{d}t\right] &\le \mathrm{e}^{\varepsilon l(0)+1}\left(\mathbb{E}^\xi \mathrm{e}^{2\varepsilon^2 \langle \mathscr{M}\rangle(T)}\right)^{1/2} \\ &\le \mathrm{e}^{1+\varepsilon l(0)}\left(\mathbb{E}^\xi \mathrm{e}^{2\varepsilon^2\|\sigma\|^2\delta\int_0^T \bar{l}_t\mathrm{d}t}\right)^{1/2} = \mathrm{e}^{1+\varepsilon l(0)}\left(\mathbb{E}^\xi \mathrm{e}^{\frac{\varepsilon}{T\mathrm{e}^{1+cT}}\int_0^T \bar{l}_t\mathrm{d}t}\right)^{1/2}.\end{aligned}$$

By using stopping times as in the proof of Lemma 2.1 we may assume that

$$\mathbb{E}^\xi \exp\left[\frac{\varepsilon}{T\mathrm{e}^{1+cT}}\int_0^T \bar{l}_t \mathrm{d}t\right] < \infty$$

so that

$$\mathbb{E}^\xi \exp\left[\frac{\varepsilon}{T\mathrm{e}^{1+cT}}\int_0^T \bar{l}_t \mathrm{d}t\right] \le \mathrm{e}^{2+2\varepsilon l(0)}.$$

This completes the proof by noting that

$$\frac{1}{2\|\sigma\|^2\delta T^2\mathrm{e}^{2+2cT}}\int_0^T \|W(X_t,Y_t)\|_\infty \mathrm{d}t \le \frac{r_0\|W(\xi)\|_\infty}{2\|\sigma\|^2\delta T^2 \mathrm{e}^{2+2cT}} + \frac{\varepsilon}{T\mathrm{e}^{1+cT}}\int_0^T \bar{l}_t \mathrm{d}t.$$

□

Proof of Corollary 1.3. Let v and α be given in Corollary 1.2. By the semigroup property and the Jensen inequality, we will only consider $T - r_0 \in (0,1]$.

(1) By (1.5) and the definitions of α and v, there exists a constant $C > 0$ such that

NNO

$$\begin{aligned}&|v'(s)h_2(0) + \alpha'(s)| \le C 1_{[0,T-r_0]}(s)|h(0)|\Big(\frac{1}{T-r_0} + \frac{\|M\|}{(T-r_0)^{2(k+1)}}\Big), \quad s\in[0,T],\\ &|\Theta(s)| \le C|h(0)|\Big(1 + \frac{\|M\|}{(T-r_0)^{2k+1}}\Big), \quad s\in[0,T],\\ &\|\Theta_s\|_\infty \le C\Big(\|h\|_\infty + \frac{\|M\|\cdot|h(0)|}{(T-r_0)^{2k+1}}\Big), \quad s\in[0,T].\end{aligned} \tag{3.1}$$

Therefore, it follows from $(A3)$ and $(A4)$ that

NN (3.2)
$$\begin{aligned}|N(s)| \leq & C1_{[0,T-r_0]}(s)|h(0)|\Big(\frac{1}{T-r_0}+\frac{\|M\|}{(T-r_0)^{2(k+1)}}\Big)\\ &+C\Big(\|h\|_\infty+\frac{\|M\|\cdot|h(0)|}{(T-r_0)^{2k+1}}\Big)\|W(X_s,Y_s)\|_\infty^l\end{aligned}\tag{3.2}$$

holds for some constant $C>0$. Combining this with Theorem 1.1 we obtain

$$\begin{aligned}|\nabla_h P_T f(\xi)| &\leq C\sqrt{P_T f^2(\xi)}\Big(\mathbb{E}^\xi\int_0^T |N(s)|^2\mathrm{d}s\Big)^{1/2}\\ &\leq C\sqrt{P_T f^2(\xi)}\Big\{|h(0)|\Big(1+\frac{\|M\|}{(T-r_0)^{2k+1}}\Big)\\ &\quad+\Big(\|h\|_\infty+\frac{\|M\|\cdot|h(0)|}{(T-r_0)^{2k+1}}\Big)\Big(\int_0^T\mathbb{E}^\xi\|W(X_s,Y_s)\|_\infty^{2l}\mathrm{d}s\Big)^{1/2}\Big\},\end{aligned}$$

This completes the proof of (1) since due to Lemma 2.1 one has

$$\mathbb{E}^\xi\|W(X_s,Y_s)\|_\infty^{2l}\leq 3\|W(\xi)\|_\infty^{2l}\mathrm{e}^{Cs},\quad s\in[0,T]$$

for some constant $C>0$.

(2) By Theorem 1.1 and the Young inequality (cf. [2, Lemma 2.4]), we have

H1 (3.3)
$$\begin{aligned}|\nabla_h P_T f|(\xi)\leq & r\big\{P_T f\log f-(P_T f)\log P_T f\big\}(\xi)\\ &+rP_T f(\xi)\log\mathbb{E}^\xi\mathrm{e}^{\frac{1}{r}\int_0^T\langle N(s),(\sigma^*)^{-1}\mathrm{d}B(s)\rangle},\quad r>0.\end{aligned}\tag{3.3}$$

Next, it follows from (3.2) that

H2 (3.4)
$$\begin{aligned}&\Big(\mathbb{E}^\xi\exp\Big[\frac{1}{r}\int_0^T\langle N(s),(\sigma^*)^{-1}\mathrm{d}B(s)\rangle\Big]\Big)^2\leq\mathbb{E}^\xi\exp\Big[\frac{2\|\sigma^{-1}\|^2}{r^2}\int_0^T|N(s)|^2\mathrm{d}s\Big]\\ &\leq\exp\Big[\frac{C_1|h(0)|^2}{r^2}\Big(\frac{1}{T-r_0}+\frac{\|M\|^2}{(T-r_0)^{4k+3}}\Big)\Big]\\ &\quad\times\mathbb{E}^\xi\exp\Big[\frac{C_1}{r^2}\Big(\|h\|_\infty^2+\frac{\|M\|^2|h(0)|^2}{(T-r_0)^{4k+2}}\Big)\int_0^T\|W(X_s,Y_s)\|_\infty^{2l}\mathrm{d}s\Big],\quad T\in(r_0,1+r_0]\end{aligned}\tag{3.4}$$

holds for some constant $C_1\in(0,\infty)$. Since $2l\in[0,1)$ and $T\leq 1+r_0$, there exists a constant $C_2\in(0,\infty)$ such that

$$\beta\|W(X_s,Y_s)\|_\infty^{2l}\leq\frac{(\frac{\|h\|_\infty^2}{r^2}\wedge 1)\|W(X_s,Y_s)\|_\infty}{2\|\sigma\|^2\delta T^2\mathrm{e}^{2+2cT}}+C_2\beta^{\frac{1}{1-2l}}\Big(\frac{\|h\|_\infty^2}{r^2}\wedge 1\Big)^{-\frac{2l}{1-2l}},\ \beta>0.$$

Taking

$$\beta = \frac{C_1}{r^2}\bigg(\|h\|_\infty^2 + \frac{\|M\|^2|h(0)|^2}{(T-r_0)^{4k+2}}\bigg),$$

and applying Corollary 3.2, we arrive at

$$\begin{aligned}
\mathbb{E}^\xi \exp\bigg[\beta\int_0^T \|W(X_s,Y_s)\|_\infty^{2l}\mathrm{d}s\bigg] &\le \exp\bigg[C_2\beta^{\frac{1}{1-2l}}\Big(\frac{\|h\|_\infty^2}{r^2}\wedge 1\Big)^{-\frac{2l}{1-2l}}\bigg]\\
&\quad\times\bigg(\mathbb{E}^\xi\exp\bigg[\frac{1}{2\|\sigma\|^2\delta T^2\mathrm{e}^{2+2cT}}\int_0^T\|W(X_s,Y_s)\|_\infty\mathrm{d}s\bigg]\bigg)^{\frac{\|h\|_\infty^2}{r^2}\wedge 1}\\
&\le \exp\bigg[\frac{C_3}{r^2}\bigg\{\|h\|_\infty^2\|W(\xi)\|_\infty + \Big(\|h\|_\infty^2+\frac{\|M\|^2|h(0)|^2}{(T-r_0)^{4k+2}}\Big)^{\frac{1}{1-2l}}\Big(\frac{r^2}{\|h\|_\infty^2}\vee 1\Big)^{\frac{2l}{1-2l}}\bigg\}\bigg]
\end{aligned}$$

for some constant $C_3\in(0,\infty)$ and all $T\in(r_0,1+r_0]$. Therefore, the desired entropy-gradient estimate follows by combining this with (3.3) and (3.4).

(3) Let $C'>0$ be such that $r\ge C'\Big(\|h\|_\infty+\frac{\|M\|\cdot|h(0)|}{(T-r_0)^{2k+1}}\Big)$ implies

$$\frac{C_1}{r^2}\Big(\|h\|_\infty^2+\frac{\|M\|^2|h(0)|^2}{(T-r_0)^{4k+2}}\Big)\le\frac{1}{2\|\sigma\|^2\delta T^2\mathrm{e}^{2+2cT}},$$

so that by Corollary 3.2

$$\begin{aligned}
&\mathbb{E}^\xi\exp\bigg[\frac{C_1}{r^2}\Big(\|h\|_\infty^2+\frac{\|M\|^2|h(0)|^2}{(T-r_0)^{4k+2}}\Big)\int_0^T\|W(X_s,Y_s)\|_\infty^{2l}\mathrm{d}s\bigg]\\
&\le\bigg(\mathbb{E}^\xi\exp\bigg[\frac{1}{2\|\sigma\|^2\delta T^2\mathrm{e}^{2+2cT}}\int_0^T\|W(X_s,Y_s)\|_\infty\mathrm{d}s\bigg]\bigg)^{\frac{2C_1\|\sigma\|^2\delta T^2\mathrm{e}^{2+2cT}}{r^2}\Big(\|h\|_\infty^2+\frac{\|M\|^2|h(0)|^2}{(T-r_0)^{4k+2}}\Big)}\\
&\le\exp\bigg[\frac{C\|W(\xi)\|_\infty}{r^2}\Big(\|h\|_\infty^2+\frac{\|M\|^2|h(0)|^2}{(T-r_0)^{4k+2}}\Big)\bigg]
\end{aligned}$$

holds for some constant $C>0$. Then proof is finished by combining this with (3.3) and (3.4). □

Proof of Theorem 1.5. Again, we only prove for $T\in(r_0,1+r_0]$. Applying (2.9) to $\varepsilon=1$ and using $\log f$ to replace f, we obtain

W0 (3.5)
$$P_T\log f(\xi+h)=\mathbb{E}\{R^1(T)\log f(X_T,Y_T)\}\le\log P_Tf(\xi)+\mathbb{E}(R^1\log R^1)(T).$$

Next, taking $\varepsilon=1$ in (2.6) and letting $n\uparrow\infty$, we arrive at

W1 (3.6)
$$\mathbb{E}(R^1\log R^1)(T)\le\frac{1}{2}\mathbb{E}_{\mathbb{Q}_1}\int_0^T|\sigma^{-1}\Phi^1(r)|^2\mathrm{d}r.$$

By $(A3')$, $(A4')$, (3.1) and the definition of Φ^1, we have

$$
\begin{aligned}
|\sigma^{-1}\Phi^1(s)|^2 \le & C_1\Big\{\|W(X_s^1,Y_s^1)\|_\infty^{2l}+U^2\Big(C_1\|h\|_\infty+\frac{C_1\|M\|\cdot|h(0)|}{(T-r_0)^{2k+1}}\Big)\Big\}\|h\|_\infty^2\\
&+C_1|h(0)|^2\Big(\frac{1}{(T-r_0)^2}+\frac{\|M\|^2}{(T-r_0)^{4(k+1)}}\Big)1_{[0,T-r_0]}(s)
\end{aligned}
$$

for some constant $C_1>0$. Then the proof is completed by combining this with (3.5), (3.6) and Lemma 2.1 (note that $(X^1(s),Y^1(s))$ under $\mathbb{Q}_1$ solves the same equation as (X_s,Y_s) under $\mathbb{P}$). □

4 Discrete Time Delay Case and Examples

In this section we first present a simple example to illustrate our main results presented in Section 1, then relax assumption **(A)** for the discrete time delay case in order to cover some highly non-linear examples.

Example 4.1. For $\alpha\in C([-r_0,0];\mathbb{R})$, consider functional SDE on $\mathbb{R}^2$

$$
\begin{cases}\mathrm{d}X(t)=-\{X(t)+Y(t)\}\mathrm{d}t\\ \mathrm{d}Y(t)=\mathrm{d}B(t)+\Big\{-\varepsilon Y^3(t)+Y(t-r_0)+\int_{-r_0}^0\alpha(\theta)X(t+\theta)\mathrm{d}\theta\Big\}\mathrm{d}t\end{cases}\tag{4.1}
$$

with initial data $\xi=(\xi_1,\xi_2)\in C([-r_0,0];\mathbb{R}^2)$, where $\varepsilon\ge0$ and $n\in\mathbb{N}$ are constants. For $z=(x,y)\in\mathbb{R}^2$, let $W(x,y)=1+|x|^2+|y|^2$ and set $Z(z)=-y^3$ and $b(\xi)=\int_{-r_0}^0\alpha(\theta)\xi_1(\theta)\mathrm{d}\theta+\xi_2(-r_0)$. By a straightforward computation one has for $x,y\in\mathbb{R}$

$$LW(x,y)=1-2x(x+y)-2\varepsilon y^{2n}\le 3W(x,y)$$

and for $\xi\in C([-r_0,0];\mathbb{R}^2)$

$$
\begin{aligned}
\langle b(\xi),\nabla^{(2)}W(\xi(0))\rangle &\le 2\Big|\int_{-r_0}^0\alpha(\theta)\xi_1(\theta)\mathrm{d}\theta+\xi_2(-r_0)\Big||\xi_2(0)|\\
&\le 2\Big(1+\int_{-r_0}^0\alpha(\theta)\mathrm{d}\theta\Big)\|\xi\|_\infty^2.
\end{aligned}
$$

Then conditions (A1) and (A2) hold. Next, there exists a constant $c>0$ such that for any $z=(x,y)$ and $z'=(x',y')\in\mathbb{R}^2$,

$$|Z(z)-Z(z')|=\varepsilon|y^3-y'^3|\le c|y-y'|(|y'|^2+|y-y'|^2).$$

Finally, for $\xi=(\xi_1,\xi_2),\xi'=(\xi_1',\xi_2')\in C([-r_0,0];\mathbb{R}^2)$,

$$|b(\xi)-b(\xi')|\le\sqrt{2}\Big(\int_{-r_0}^0|\alpha(\theta)|\mathrm{d}\theta\vee1\Big)\|\xi-\xi'\|_\infty.$$

So, $(A3)$ holds for $l=1$ whenever $|y-y'|\le1$ and $(A4)$ holds for any $l\ge0$. Moreover, $(A3')$ and $(A4')$ hold for $U(|z|)=|z|^2,z\in\mathbb{R}^2$. Therefore, Theorem 1.1, Theorem 1.5 and Corollary 1.3 hold.

To derive the entropy-gradient estimate and the Harnack inequality as in Corollary 1.4, we need to weaken the assumption **(A)**. To this end, we consider a simpler setting where the delay is time discrete. Consider

E20
$$\begin{cases} \mathrm{d}X(t) = \{AX(t) + MY(t)\}\mathrm{d}t, \\ \mathrm{d}Y(t) = Z(X(t), Y(t)) + \tilde{b}(X(t-r_0), Y(t-r_0))\mathrm{d}t + \sigma \mathrm{d}B(t), \end{cases} \tag{4.2}$$

with initial data $\xi \in \mathscr{C}$, where $Z, \tilde{b} : \mathbb{R}^{m+d} \to \mathbb{R}^d$. If we define $b(\xi) = \tilde{b}(\xi(-r_0))$ for $\xi = (\xi_1, \xi_2) \in \mathscr{C}$, then equation (4.2) can be written as equation (1.1). For $(x, y), (x', y') \in \mathbb{R}^{m+d}$, define the diffusion operator associated with (4.2) by

$$\mathscr{L}W(x, y; x', y') = LW(x, y) + \langle \tilde{b}(x', y'), \nabla^{(2)} W(x, y)\rangle.$$

T4.2 **Theorem 4.2.** *Assume that there exist constants $\alpha, \beta, \gamma > 0$ with $\beta \geq \gamma$, functions $W \in C^2(\mathbb{R}^{m+d})$ with $W \geq 1$ and $U \in C(\mathbb{R}^{m+d}; \mathbb{R}_+)$ such that for $(x, y), (x', y') \in \mathbb{R}^{m+d}$*

E21
$$\mathscr{L}W(x, y; x', y') \leq \alpha\{W(x, y) + W(x', y')\} - \beta U(x, y) + \gamma U(x', y'). \tag{4.3}$$

Assume further that there exists $\nu > 0$ such that for $z = (x, y), z' = (x', y') \in \mathbb{R}^{m+d}$ with $|z - z'| \leq 1$

E25
$$|Z(z) - Z(z')|^2 \vee |\tilde{b}(z) - \tilde{b}(z')|^2 \leq \nu |z - z'|^2 W(z'). \tag{4.4}$$

Then for $\delta := (\alpha r_0 + 1)\|W(\xi)\|_\infty + \gamma r_0 \|U(\xi)\|_\infty$ and $t \geq 0$

E24
$$\mathbb{E}^\xi W(X(t), Y(t)) \leq \delta e^{2\alpha t}, \tag{4.5}$$

and

E23
$$\begin{aligned} |\nabla_h P_T f(\xi)| \leq C\sqrt{P_T f^2(\xi)} \bigg\{ &|h(0)| \Big(1 + \frac{\|M\|}{(T - r_0)^{2k+1} \wedge 1}\Big) + r_0^{\frac{1}{2}} \|W(\xi)\|_\infty^{\frac{1}{2}} \|h\|_\infty \\ &+ |h(0)| \sqrt{\delta(T \wedge (1 + r_0))} \Big(1 + \frac{\|M\|}{(T - r_0)^{2k+1}}\Big) \bigg\} \end{aligned} \tag{4.6}$$

for all $T > r_0, \xi, h \in \mathscr{C}$ and $f \in \mathscr{B}_b(\mathscr{C})$, where $C > 0$ is some constant. If moreover there exist constants $K, \lambda_i \geq 0, i = 1, 2, 3, 4$, with $\lambda_1 \geq \lambda_2$ and $\lambda_3 \geq \lambda_4$, functions $\tilde{W} \in C^2(\mathbb{R}^{m+d})$ with $\tilde{W} \geq 1$ and $\tilde{U} \in C(\mathbb{R}^{m+d}; \mathbb{R}_+)$ such that for $(x, y), (x', y') \in \mathbb{R}^{m+d}$

E28
$$\frac{\mathscr{L}\tilde{W}(x, y; x', y')}{\tilde{W}(x, y)} \leq K - \lambda_1 W(x, y) + \lambda_2 W(x', y') - \lambda_3 \tilde{U}(x, y) + \lambda_4 \tilde{U}(x', y'), \tag{4.7}$$

then there exist constants $\delta_0, C > 0$ such that for $r \geq \delta_0/(T - r_0)^{2k+1}, \xi, h \in \mathscr{C}$ and positive $f \in \mathscr{B}_b(\mathscr{C})$

E26
$$\begin{aligned} |\nabla_h P_T f|(\xi) &\leq r\big\{P_T f \log f - (P_T f) \log P_T f\big\}(\xi) \\ &+ \frac{C P_T f}{2r} \bigg\{ |h(0)|^2 \Big(\frac{1}{(T - r_0) \wedge 1} + \frac{\|M\|^2}{\{(T - r_0) \wedge 1\}^{4k+3}} \Big) \\ &+ \frac{(1 + \|M\|^2)|h(0)|^2}{\{(T - r_0) \wedge 1\}^{4k+2}} \Big(\lambda_2 r_0 \|W(\xi)\|_\infty + \lambda_4 r_0 \|\tilde{U}(\xi)\|_\infty + KT + \log \tilde{W}(\xi(0)) \Big) \bigg\}. \end{aligned} \tag{4.8}$$

Proof. By the Itô formula one has for any $t \geq 0$

$$\begin{aligned}
\mathbb{E}^\xi W(X(t),Y(t)) &\leq W(\xi(0)) + \alpha \mathbb{E}^\xi \int_0^t \{W(X(s),Y(s)) + W(X(s-r_0),Y(s-r_0))\}\mathrm{d}s \\
&\quad - \beta \mathbb{E}^\xi \int_0^t U(X(s),Y(s))ds + \gamma \mathbb{E}^\xi \int_0^t U(X(s-r_0),Y(s-r_0))ds \\
&\leq W(\xi(0)) + \alpha \int_{-r_0}^0 W(X(s),Y(s))ds + \gamma \int_{-r_0}^0 U(X(s),Y(s))ds \\
&\quad + 2\alpha \mathbb{E}^\xi \int_0^t W(X(s),Y(s))ds \\
&\leq \delta + 2\alpha \mathbb{E}^\xi \int_0^t W(X(s),Y(s)ds.
\end{aligned}$$

Then (4.5) follows from the Gronwall inequality.

By Theorem 1.1, for $T - r_0 \in (0,1]$ and some $C > 0$ we can deduce that

$$|\nabla_h P_T f(\xi)| \leq C\sqrt{P_T f^2(\xi)} \Big(\mathbb{E}^\xi \int_0^T |N(s)|^2 \mathrm{d}s\Big)^{1/2},$$

where for $s \in [0,T]$

$$N(s) := (\nabla_{\Theta(s)} Z)(X(s),Y(s)) + (\nabla_{\Theta(s-r_0)} \tilde{b})(X(s-r_0),Y(s-r_0)) - v'(s)h_2(0) - \alpha'(s).$$

Recalling the first two inequalities in (3.1) and combining (4.4) yields that for some $C > 0$

$$\begin{aligned}
|\nabla_h P_T f(\xi)| &\leq C\sqrt{P_T f^2(\xi)} \bigg\{ \Big(\int_0^T |v'(s)h_2(0) + \alpha'(s)|^2 ds \Big)^{1/2} \\
&\quad + \Big(\mathbb{E}^\xi \int_0^T |\Theta(s)|^2 W(X(s),Y(s))\mathrm{d}s \Big)^{1/2} \\
&\quad + \Big(\mathbb{E}^\xi \int_0^T |\Theta(s-r_0)|^2 W(X(s-r_0),Y(s-r_0))\mathrm{d}s \Big)^{1/2} \bigg\} \\
&\leq C\sqrt{P_T f^2(\xi)} \bigg\{ |h(0)| \Big(1 + \frac{\|M\|}{(T-r_0)^{2k+1}}\Big) + r_0^{\frac{1}{2}} \|W(\xi)\|_\infty^{\frac{1}{2}} \|h\|_\infty \\
&\quad + |h(0)| \Big(1 + \frac{\|M\|}{(T-r_0)^{2k+1}}\Big) \Big(\int_0^T \mathbb{E}^\xi W(X(s),Y(s))\mathrm{d}s \Big)^{1/2} \bigg\}.
\end{aligned}$$

This, together with (4.5), leads to (4.6).

Due to (3.3) and (3.4) we can deduce that there exists $C > 0$ such that for arbitrary $r > 0$ and $T - r_0 \in (0,1]$

E30

$$\begin{aligned}
|\nabla_h P_T f|(\xi) &\leq r\big\{P_T f \log f - (P_T f)\log P_T f\big\}(\xi) \\
&\quad + \frac{r P_T f(\xi)}{2} \bigg\{ \frac{C|h(0)|^2}{r^2} \Big(\frac{1}{T-r_0} + \frac{\|M\|^2}{(T-r_0)^{4k+3}} \Big) + \frac{C\|h\|_\infty^2 \|W(\xi)\|_\infty r_0}{r^2} \\
&\quad + \log \mathbb{E}^\xi \exp\Big[\frac{C(1+\|M\|^2)|h(0)|^2}{r^2 (T-r_0)^{4k+2}} \int_0^T W(X(s),Y(s))\mathrm{d}s \Big] \bigg\}.
\end{aligned} \tag{4.9}$$

Moreover, since for $s \in [0,T]$

$$\tilde{W}(X(s),Y(s))\exp\Big(-\int_0^s \frac{\mathscr{L}\tilde{W}(X(r),Y(r),X(r-r_0),Y(r-r_0))}{\tilde{W}(X(r),Y(r))}\mathrm{d}r\Big)$$

is a local martingale by the Itô formula, in addition to $\tilde{W} \geq 1$, we obtain from (4.7) that

E29 (4.10)
$$\begin{aligned}
&\mathbb{E}^\xi \exp\Big[(\lambda_1-\lambda_2)\int_0^T W(X(s),Y(s))\mathrm{d}s - \lambda_2 r_0\|W(\xi)\|_\infty\Big]\\
&\leq \mathbb{E}^\xi \exp\Big[\int_0^T \Big(\lambda_1 W(X(s),Y(s)) - \lambda_2 W(X(s-r_0),Y(s-r_0))\Big)\mathrm{d}s\Big]\\
&\leq \mathbb{E}^\xi \exp\Big[KT - \int_0^T \frac{\mathscr{L}\tilde{W}(X(s),Y(s);X(s-r_0),Y(s-r_0))}{\tilde{W}(X(s),Y(s))}\mathrm{d}s\\
&\qquad\qquad - \lambda_3\int_0^T \tilde{U}(X(s),Y(s))\mathrm{d}s + \lambda_4\int_0^T \tilde{U}(X(s-r_0),Y(s-r_0))ds\Big]\\
&\leq \exp(\lambda_4 r_0\|\tilde{U}(\xi)\|_\infty + KT)\\
&\quad\times \mathbb{E}^\xi\Big[\tilde{W}(X(T),Y(T))\exp\Big(-\int_0^T \frac{\mathscr{L}\tilde{W}(X(s),Y(s);X(s-r_0),Y(s-r_0))}{\tilde{W}(X(s),Y(s))}\mathrm{d}s\Big)\Big]\\
&\leq \exp(\lambda_4 r_0\|\tilde{U}(\xi)\|_\infty + KT)\tilde{W}(\xi(0)).
\end{aligned}$$

Combining (4.9) and (4.10), together with the Hölder inequality, yields (4.8). □

The next example shows that Theorem 4.2 applies to the equation (4.2) with a highly non-linear drift.

Ex4.2 **Example 4.3.** Consider delay SDE on $\mathbb{R}^2$

(4.11)
$$\begin{cases}\mathrm{d}X(t) = -\{X(t)+Y(t)\}\mathrm{d}t\\ \mathrm{d}Y(t) = \mathrm{d}B(t) + \Big\{-Y^3(t) + \frac{1}{4}Y^3(t-r_0) + \frac{1}{2}X(t) - Y(t)\Big\}\mathrm{d}t\end{cases}$$

with initial data $\xi \in C([-r_0,0];\mathbb{R}^2)$. In this example for $z=(x,y), z'=(x',y') \in \mathbb{R}^2$ let $Z(z) = \frac{1}{2}x - y - y^3$ and $b(z') = \frac{1}{4}y'^3$. For $W(x,y) = 1 + x^2 + y^4$ it is easy to see that

$$\begin{aligned}
\mathscr{L}W(x,y;x',y') &= -2x(x+y) + 4y^3\Big(\frac{1}{2}x - y - y^3 + \frac{1}{4}y'^3\Big)\\
&\leq -x^2 + y^2 - 4y^4 - 4y^6 + y^3y'^3 + 2y^3x\\
&\leq y^2 - 4y^4 - \frac{5}{2}y^6 + \frac{1}{2}y'^6.
\end{aligned}$$

Then (4.3) holds for $\beta = \frac{5}{2}, \gamma = \frac{1}{2}$ and $U(x,y) = y^6$. Moreover for $z=(x,y), z'=(x',y') \in \mathbb{R}^2$ there exists $c>0$ such that

$$|Z(z)-Z(z')|^2 \vee |b(z)-b(z')|^2 \leq c|z-z'|^2(|y-y'|^4 + |y'|^4).$$

Thus condition (4.4) holds, Therefore, by Theorem 4.2 we obtain (4.6).

To derive (4.8), we take $w(x,y) = \frac{1}{4}(x^2+y^4)+\frac{1}{10}xy$ and set $\tilde{W}(x,y) = \exp(w(x,y)-\inf w)$. Compute for $(x,y,x',y') \in \mathbb{R}^4$

$$\begin{aligned}
\frac{\mathscr{L}\tilde{W}}{\tilde{W}}(x,y,x',y') &= \mathscr{L}\log\tilde{W}(x,y) + \frac{1}{2}|\partial_y \log\tilde{W}|^2(x,y) \\
&\le -\Big(\frac{1}{2}x + \frac{1}{10}y\Big)(x+y) + \Big(y^3 + \frac{1}{10}x\Big)\Big(\frac{1}{2}x - y - y^3 + \frac{1}{4}y'^3\Big) + \frac{3}{2}y^2 \\
&\quad + \frac{1}{2}\Big(y^3 + \frac{1}{10}x\Big)^2 \\
&\le 0.5((0.35)^2/\epsilon + 1.4)^2 - (0.2325-\epsilon)x^2 - 0.5y^4 - 0.175y^6 + 0.1375y'^6,
\end{aligned}$$

where $\epsilon > 0$ is some constant such that $0.2325-\epsilon > 0$. Then condition (4.7) holds. Therefore, by Theorem 4.2 we obtain (4.8), which implies the Harnack inequality as in Corollary 1.4 according to [5, Proposition 4.1].

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